

# Matter Outside a Static Higher-Dimensional Black Hole

Marek Rogatko

*Institute of Physics*

*Maria Curie-Sklodowska University*

*20-031 Lublin, pl. Marii Curie-Sklodowskiej 1, Poland*

*marek.rogat@poczta.umcs.lublin.pl*

*rogat@kft.umcs.lublin.pl*

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We considered matter fields composed of a perfect fluid in the static higher-dimensional spherically symmetric asymptotically flat black hole spacetime. The proof of the nonexistence of perfect fluid matter in such background was provided under the auxiliary condition, which can be interpreted as a relation connecting stellar mass and black hole mass in question.

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## I. INTRODUCTION

One of the most important issues of gravity theory are connected with a gravitational collapse and the emergence of black holes. In the case of general relativity the uniqueness theorem for black holes states that the static electrovac black hole solution is diffeomorphic to the domain of outer communication of Reissner-Nordström (RN) spacetime whereas the rotating one is diffeomorphic to Kerr-Newman (KN) spacetime. The first classification of non-singular black hole solutions was presented in Refs. [1]-[8]. The condition of non-degeneracy of the event horizon in the proofs of the uniqueness theorem was eliminated in [9, 10]. In the case of static electro-vacuum black holes with degenerate components of the event horizons, the study of near-horizon geometry enables to finish the classification [11].

The problem of stationary axisymmetric black hole solutions was far more complicated. It was treated in Refs.[12]-[14], while the complete proof was found by Mazur [15] and Bunting [16]-[17] (for a review of the uniqueness of black hole solutions see, e.g., [18] and references therein).

It turned out that M/string theory attempts of unifications of all known forces of Nature also triggered works bounded with the mathematical aspects of black holes in the low-energy string theory as well as its higher dimensional generalization. Various aspects of the low-energy string black holes including the staticity theorem, uniqueness theorems in Einstein-Maxwell axion dilaton (EMAD) gravity, dilaton gravity with auxiliary  $U(1)$ -gauge fields and supergravity theories were widely treated [19].

The uniqueness theorem for static  $n$ -dimensional black hole, both in vacuum and charged case is well established [20]. The complete classification of  $n$ -dimensional charged black holes having both degenerate and non-degenerate components of event horizon was presented in Refs.[21]. In Ref.[22], taking into account the both *electric* and *magnetic* components of  $(n-2)$ -gauge form  $F_{\mu_1 \dots \mu_{n-2}}$  the uniqueness of static higher dimensional *electrically* and *magnetically* charged black hole containing an asymptotically flat hypersurface with compact interior and non-degenerate components of the event horizon was proved. On the other hand, the staticity theorem for generalized Einstein-Maxwell (EM) system was discussed in [23].

Uniqueness theorem for stationary axisymmetric  $n$ -dimensional black holes is much more delicate problem. It turns out that generalization of Kerr metric to arbitrary  $n$ -dimensions proposed by Myers-Perry [24] is not unique. Rotating black hole ring solution with the same angular momentum and mass but the horizon homeomorphic to  $S^2 \times S^1$  presented in [25] constitutes the counterexample. However, the existence of black rings is consistent with the generalization of Hawking's theorem [26]. Namely, as was shown in [27, 28] cross sections of event horizons and outer horizons were of the positive Yamabe type. They admit metrics of positive scalar curvature.

Recent results concerning the attempts of treating the problem of uniqueness of stationary axisymmetric higher dimensional black objects (black holes and black rings) were presented in Refs.[29].

The other tantalizing question in mathematical theory of black holes is concerned with the uniqueness problem for stationary or static black holes and matter. Depending on the matter model in question, black holes may allow to exist the nontrivial fields outside its event horizon. This question was elaborated in Refs.[30]. It was also revealed [31] RN black hole solution with both an electric and magnetic charges can be destroyed in the presence of a massless Dirac fermion field. On the other hand, it was shown [32] that the only black hole solutions of four-dimensional spinor Einstein-dilaton-Yang-Mills field equations of motion were those for which spinors vanished identically outside black hole. It means that Dirac fermion fields either enter the black hole in question or escape to infinity. This tendency was also confirmed in Ref.[33].

On the other hand, the proof that the stellar model composed of perfect fluid, nonrotating, self-gravitating and

physically isolated is spherical symmetric has also a long story. Significant progress was made in Refs.[34]-[37]. Among all, it was shown that the spatial part of the underlying geometry of static stellar model ought to be conformally flat, due to the consequence of the positive mass theorem. Because of the fact that spatial conformal flatness is equivalent to the spherical symmetry in static stellar model [38], the above arguments completed the proof. However, in Ref.[39] various technical assumptions were get rid of and unphysical restrictions on the equation of state of perfect fluid were weaken. Recently, it was shown [40] that matter configuration composed of a perfect fluid could not be at rest outside a four-dimensional black hole in asymptotically flat static spacetime.

The main motivation for the present work stems from the fact that many contemporary theories of unifications of the all known forces in Nature like superstring/M-theory require extra dimensions to be consistently formulated. The theoretical properties of the higher-dimensional gravitational field could be quite different. The appearance of black rings, black Satellites and black lens objects confirms this conjecture. So it will be of much interest to get an insight into the multidimensional physics which may lead to a wider understanding of the gravitational field. Just, motivated by higher dimensional gravity theories which are crucial ingredients in modern unification schemes, we shall study the problem of the uniqueness of static  $n$ -dimensional black hole and a perfect fluid matter outside the black hole event horizon. We provide a proof for the nonexistence of a perfect fluid star outside the black hole event horizon, under the condition that star mass is smaller comparing to the black hole one.

Our paper is organized as follows. In Sec.II we describe higher dimensional Einstein perfect fluid system and introduce the quantities which are needed in the proof. In Sec.III we introduce the conformal transformation of the spatial part of the line element in question and establish important properties of it. Sec.IV will be devoted to the proof of the nonexistence of a perfect fluid matter in static,  $n$ -dimensional black hole background. In Sec.V we provide the physical meaning of the inequality which is crucial in the uniqueness proof. Then, one concludes the investigations.

## II. HIGHER DIMENSIONAL EINSTEIN PERFECT FLUID SYSTEM

In this section we shall consider higher dimensional Einstein field equations with perfect fluid matter which is in static equilibrium. A static spacetime admits a hypersurface orthogonal to asymptotically timelike Killing vector field  $k_\alpha$ . Suppose that  $t$  will be a function which level surface is orthogonal to  $k_\alpha$  and  $t^\beta \partial_\beta t = 1$ . Thus, the line element of  $n$ -dimensional static spacetime with the asymptotically timelike Killing vector field  $k_\alpha = (\frac{\partial}{\partial t})_\alpha$  and  $V^2 = -k_\mu k^\mu$  is subject to the relation

$$ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j, \quad (1)$$

where  $g_{ij}$  is an induced metric on the hypersurface orthogonal to  $k_\alpha$ , i.e., the metric on  $t = \text{const}$  submanifold.  $V$  and  $g_{ij}$  are independent of the  $t$ -coordinate as the quantities of the hypersurface  $\Sigma$  of constant  $t$ .

The Einstein perfect fluid equations of motion are provided by

$$D_a D^a V = \frac{8\pi V}{n-2} \left[ (n-3) \rho + (n-1) p \right], \quad (2)$$

$${}^{(g)}R_{ij} = \frac{8\pi g_{ij}}{n-2} \left( \rho - p \right) + \frac{1}{V} D_i D_j V, \quad (3)$$

$$D_c p = -\frac{D_c V}{V} \left( \rho + p \right), \quad (4)$$

where  $D_i$  and  ${}^{(g)}R_{ij}$  denote covariant derivative and Ricci curvature tensor  ${}^{(g)}R_{ij}$  on  $(\Sigma, g_{ij})$ , respectively. In addition there is a perfect fluid matter characterized by the energy density  $\rho$  and pressure  $p$  in a spacetime in question. The perfect fluid fulfills an equation of state of the form  $p = p(\rho)$ . On the other hand, the surface of a perfect fluid star is a  $(n-2)$ -dimensional closed, connected equipotential surface for which  $V(x) = V_S > 0$ .

As in Ref.[24] we define mass of the higher dimensional object as

$$M = \frac{(n-2) \Omega_{n-2}}{8\pi} m_\mu, \quad (5)$$

where  $\Omega_{n-2}$  is the volume of  $(n-2)$ -dimensional unit sphere. On this account, we have

$$M = \int_0^r \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} r_\mu^{n-2} \rho dr = \frac{(n-2) \Omega_{n-2}}{8\pi} m_\mu. \quad (6)$$

According to the above relation one gets that

$$m_\mu = \frac{8 \pi}{(n-1)(n-2)} \rho r_\mu^{n-1}. \quad (7)$$

In the asymptotic flat spacetime we can find an appropriate coordinate system which provides the following asymptotical behaviour of  $V$ :

$$V = 1 - \frac{m_\mu}{r^{n-3}} + \mathcal{O}\left(\frac{1}{r^{n-2}}\right), \quad (8)$$

and accordingly for the metric tensor we obtain the relation

$$g_{ij} = \left(1 + \frac{2 m_\mu}{r^{n-3}}\right) \delta_{ij} + \mathcal{O}\left(\frac{1}{r^{n-2}}\right). \quad (9)$$

As was pointed out in four dimensional treatment of this subject, equation (4) provides the fact that the surface  $\rho = \text{const}$  is identical with the surface  $V = \text{const}$ . Moreover, this relation indicates that  $D_i p$  diverges at the black hole event horizon, where one has that  $V = 0$ . In the case of a black hole, the hypersurface  $\Sigma$  is a simply connected spacelike hypersurface and  $\bar{\Sigma}$  denotes the closure of it. The topological boundary of  $\Sigma$ ,  $\partial\Sigma = \bar{\Sigma} \setminus \Sigma$  is a nonempty topological manifold with  $g_{ij} k^i k^j = 0$  on  $\partial\Sigma$ , which constitutes black hole event horizon. In a static spacetime it coincides with the Killing horizon.

Due to the fact of  $D_i p$  divergence on  $V = 0$ , which constitutes an unphysical situation, we shall consider the case where stellar surface will be disjoint from black hole event horizon.

The other argument about disjointness of the star from the black hole event horizon can be provided by the theory of elliptic differential equations. Because of the fact that equation (2) is of elliptic type, the standard boundary elliptic estimate provides that near the event horizon, the norm of the gradient of  $V$  is bounded by the norm of the right-hand side of equation (2) and that of  $V$  [41]. The surface of the star is the level set  $V = V_S > 0$ , while the horizon is zero set of  $V$ . Just the gradient bound for  $V$  gives a positive lower bound for the distance between star and horizon level sets. This in turn yields that the surface of the star is disjoint from the black hole event horizon.

Because of the fact that  $V = \text{const}$  surface is identical with  $\rho = \text{const}$  surface and having in mind equation of state where  $p$  is function of density, one can regard both pressure and density as functions of  $V$ . The minimum value of  $V$  takes place inside the considered star. To proceed further we can define functions  $r_\mu(V)$  and  $m_\mu(V)$ , which are solutions of the following differential equations:

$$\frac{dr_\mu}{dV} = \frac{r_\mu \left( r_\mu^{n-3} - 2 m_\mu \right)}{V(r) \left[ \frac{8\pi}{n-2} p r_\mu^{n-1} + m_\mu (n-3) \right]}, \quad (10)$$

$$\frac{dm_\mu}{dV} = \frac{8 \pi}{(n-2)} \frac{\rho r_\mu^{n-1} \left( r_\mu^{n-3} - 2 \mu \right)}{V(r) \left[ \frac{8\pi}{n-2} p r_\mu^{n-1} + m_\mu (n-3) \right]}. \quad (11)$$

The above differential equations satisfy the boundary conditions

$$r_\mu(V_S) = R_\mu = \left( \frac{2 \mu}{1 - V_S^2} \right)^{\frac{1}{n-3}}, \quad (12)$$

$$m_\mu(V_S) = \mu, \quad (13)$$

where  $\mu$  is a constant value which can be interpreted as a local mass of the perfect fluid star in question.

Let us introduce on the domain where the solutions of equations (10) and (11) exist, the function  $W_\mu(V)$  provided by the relation of the form

$$W_\mu = \left(1 - \frac{2 m_\mu}{r_\mu^{n-3}}\right) \left(\frac{dr_\mu}{dV}\right)^{-2} = \frac{V^2 \left[ \frac{8\pi}{n-2} p r_\mu^{n-1} + m_\mu (n-3) \right]^2}{r_\mu^{n-1} \left( r_\mu^{n-3} - 2 m_\mu \right)}. \quad (14)$$

The above definition is valid inside the considered star. Consequently, outside the star it is given by the expression

$$W_\mu = \frac{(n-3)^2 \left(1 - V^2\right)^{2\left(\frac{n-2}{n-3}\right)}}{2^{\frac{2n-4}{n-3}} \mu^{\frac{2}{n-3}}}. \quad (15)$$

Hence, having in mind equation (8), the asymptotic behaviour of  $W_\mu$  near spatial infinity is subject to the relation

$$W_\mu = \frac{(n-3)^2}{\mu^{\frac{2}{n-3}}} \left[ \frac{m_\mu^{2\left(\frac{n-2}{n-3}\right)}}{r^{2(n-2)}} \right] + \mathcal{O}\left(\frac{1}{r^{2n-3}}\right). \quad (16)$$

At the surface of the star one has the following conditions:

$$\begin{aligned} r_\mu = R_\mu &> \left(2 \mu\right)^{\frac{1}{n-3}} = \left(2 m_\mu\right)^{\frac{1}{n-3}} > \\ &\left[ - \frac{16 \pi p r_\mu^{n-1}}{(n-3)(n-2)} \right]^{\frac{1}{n-3}} = 0. \end{aligned} \quad (17)$$

By virtue of the above, the derivatives  $\frac{dr_\mu}{dV}$  and  $\frac{dm_\mu}{dV}$  yield

$$\frac{dr_\mu}{dV} \Big|_{V=V_S} = \frac{R_\mu^{n-2} V_S}{m_\mu (n-3)}, \quad (18)$$

$$\frac{dm_\mu}{dV} \Big|_{V=V_S} = \frac{8 \pi}{(n-2)} \frac{\rho R_\mu^{2n-4} V_S}{[m_\mu (n-3)]}. \quad (19)$$

The above derivatives are bounded and it provides that solutions to equations (10) and (11) with these boundary conditions exist locally. From the theory of differential equations [42] one concludes that, the local existence theorem requires that left-hand sides of relations (10) and (11) need to be continuous functions of  $V$ . In other words, the local existence of functions  $r_\mu$  and  $m_\mu$  is guaranteed even if the density function  $\rho(V)$  and pressure function  $p(V)$  are not differentiable at the level surface  $V = V_S$ .

Let us consider an arbitrary level set  $V'_\mu$  in the sense that the interval  $(V'_\mu, V_S]$  constitutes the maximal interval where the solutions of (10)-(11) exist and pressure is finite  $p(V) < \infty$  as well as the following inequalities take place:

$$r_\mu > 0, \quad r_\mu^{n-3} > 2 m_\mu, \quad m_\mu > -\frac{8 \pi}{n-2} p r_\mu^{n-1}, \quad (20)$$

then having in mind the relation defining  $W_\mu$ , one achieves that  $W_\mu > 0$ , on the considered interval. As far as  $\frac{dr_\mu}{dV}$  is concerned, it is greater than zero, so  $r_\mu$  is monotonic and bounded by the relation

$$R_\mu \geq r_\mu > 0. \quad (21)$$

The same situation takes place for the other derivative in question, namely for  $\frac{dm_\mu}{dV}$ . Because of the fact that  $\frac{dm_\mu}{dV} > 0$ , then it leads to the case that  $m_\mu$  is monotonic and bounded, i.e., we arrive at the relations of the forms

$$\mu \geq m_\mu > -\frac{8 \pi}{n-2} p r_\mu^{n-1}. \quad (22)$$

Moreover, due to the equations (10) and (11), it can be readily seen that we are left with the following:

$$\frac{dm_\mu}{dr_\mu} = \frac{8 \pi}{(n-2)} \rho r_\mu^{n-2}. \quad (23)$$

Next, we assume that  $\lim_{V \rightarrow V'_\mu} p = p_\mu < \infty$ . By the direct computations it can be revealed that

$$\frac{dW_\mu}{dV} + 2 \left(n-2\right) \frac{W_\mu}{r_\mu} \frac{dr_\mu}{dV} - \frac{16 \pi V}{n-2} \left[(n-1) p + (n-3) \rho\right] = 0, \quad (24)$$

which constitutes the  $n$ -dimensional generalization of the equation derived in Ref.[39].

Equation (24) can be integrated on the interval  $(V'_\mu, V_S]$ . One attains the following result:

$$r_\mu^{2(n-2)} W_\mu = \left( \mu (n-3) \right)^2 - \int_V^{V_S} \frac{16 \pi V r_\mu^{2(n-2)}}{(n-2)} \left[ (n-1) p + (n-3) \rho \right] dV. \quad (25)$$

It can be also revealed that  $r_\mu^{2(n-2)} W_\mu$  is bounded on the considered interval. The other form of the above relation can be provided by

$$r_\mu^{2(n-2)} W_\mu = \frac{V^2 \left[ \frac{8\pi}{n-2} p r_\mu^{n-1} + m_\mu (n-3) \right]^2}{\left( 1 - \frac{2 m_\mu}{r_\mu^{n-3}} \right)}. \quad (26)$$

The expression  $r_\mu^{2(n-2)} W_\mu$  will be bounded if the numerator of Eq.(26) vanishes in the limit  $V \rightarrow V'_\mu$ . On its turn, it yields

$$\lim_{V \rightarrow V'_\mu} m_\mu (n-3) = -\frac{8\pi}{n-2} p r_\mu^{n-1}. \quad (27)$$

If  $\lim_{V \rightarrow V'_\mu} r_\mu > 0$ , then we have

$$\lim_{V \rightarrow V'_\mu} \frac{2 m_\mu}{r_\mu^{n-3}} = -\frac{16\pi}{n-2} p r_\mu^2. \quad (28)$$

On the other hand, when  $\lim_{V \rightarrow V'_\mu} r_\mu = 0$ , we arrive at the conclusion that  $\lim_{V \rightarrow V'_\mu} \frac{2 m_\mu}{r_\mu^{n-3}} = 0$ . Moreover, in other case we get  $\lim_{V \rightarrow V'_\mu} \frac{2 m_\mu}{r_\mu^{n-3}} \leq 0$ . However, this conclusion contradicts our assumption that  $\lim_{V \rightarrow V'_\mu} \frac{2 m_\mu}{r_\mu^{n-3}} = 1$ . Just, it leads us to the conclusion that

$$\sup_{(V'_\mu, V_S)} \left( \frac{2 m_\mu}{r_\mu^{n-3}} \right) < 1. \quad (29)$$

Now we shall pay attention to the behaviour of  $W_\mu$  function in the limit when  $V$  tends to  $V'_\mu$ . At the beginning we consider the case when  $\lim_{V \rightarrow V'_\mu} r_\mu = 0$ . In order to find the limit of  $W_\mu$  we arrange the function in question in useful form which implies

$$W_\mu = V^2 \left[ \frac{8\pi}{n-2} p r_\mu + \frac{m_\mu (n-3)}{r_\mu^{n-2}} \right]^2 \left( 1 - \frac{2 m_\mu}{r_\mu^{n-3}} \right)^{-1}. \quad (30)$$

At first we elaborate the expression

$$\lim_{V \rightarrow V'_\mu} \left( \frac{m_\mu (n-3)}{r_\mu^{n-2}} \right)^2, \quad (31)$$

which can be calculated using l'Hospital rule and equation (23). It turns out that the above limit is equal to zero. Next, we compute

$$\lim_{V \rightarrow V'_\mu} \left( 1 - \frac{2 m_\mu}{r_\mu^{n-3}} \right) = 1 - \lim_{V \rightarrow V'_\mu} \left( \frac{2 \frac{dm_\mu}{dr_\mu}}{(n-3) r_\mu^{n-4}} \right) = 1. \quad (32)$$

Having all the above in mind, we reach to the conclusion that  $\lim_{V \rightarrow V'_\mu} W_\mu = 0$ , which in turns yields that  $V_\mu = V'_\mu$ . Secondly, we assume that  $\lim_{V \rightarrow V'_\mu} r_\mu > 0$ . As in four-dimensional case [39],  $m_\mu$  and  $r_\mu$  are absolutely bounded in the considered case. Namely, one has that

$$\sup_{(V'_\mu, V_S)} |r_\mu| = R_\mu, \quad \sup_{(V'_\mu, V_S)} |m_\mu| < \mu + \frac{8\pi}{(n-2)} p r_\mu^{n-1}. \quad (33)$$

Now, we express the relation for  $\frac{dr_\mu}{dV}$  in the form as

$$\frac{dr_\mu}{dV} = \frac{1}{\sqrt{W_\mu}} \left( 1 - \frac{2 m_\mu}{r_\mu^{n-3}} \right)^{\frac{1}{2}}. \quad (34)$$

Because of the fact that on the interval  $(V'_\mu, V_S]$  the following relation is satisfied

$$1 > \frac{2 m_\mu}{r_\mu^{n-3}} > -\frac{16 \pi}{(n-2)} p r_\mu^2. \quad (35)$$

In effect, the bound of the  $r_\mu$  derivative is provided by

$$\left| \frac{dr_\mu}{dV} \right| < \left( \frac{1 + \frac{16 \pi}{(n-2)} p R_\mu^2}{W_\mu} \right)^{\frac{1}{2}}. \quad (36)$$

On the other hand, the bound of the derivative of  $m_\mu$  implies

$$\left| \frac{dm_\mu}{dV} \right| < \left| \frac{dm_\mu}{dr_\mu} \right| \left| \frac{dr_\mu}{dV} \right| = \frac{8 \pi}{(n-2)} \rho R_\mu^{n-2} \left( \frac{1 + \frac{16 \pi}{(n-2)} p R_\mu^2}{W_\mu} \right)^{\frac{1}{2}}. \quad (37)$$

Just, the both derivatives of  $r_\mu$  and  $m_\mu$  will be bounded at  $V = V'_\mu$  if  $\lim_{V \rightarrow V'_\mu} W_\mu > 0$ . But if it happens, it turns out that the solutions of Eqs.(10) and (11) can be extended beyond  $V = V'_\mu$  and moreover in this case  $r_\mu > 0$ ,  $r_\mu^{n-3} > 2 m_\mu$ ,  $m_\mu (n-3) > -\frac{8\pi}{(n-2)} p r_\mu^{n-1}$ , and  $p = p_\mu < \infty$  at  $V = V'_\mu$ . Using the continuity arguments one can find that these inequalities for  $r_\mu$ ,  $m_\mu$  can be extended in the vicinity of  $V'_\mu$ . But this is not the case, because it violates the assumption that  $(V'_\mu, V_S]$  is the maximal interval on which solutions of (10) and (11) can be obtained. It leads to the contradiction and one obtains that

$$\lim_{V \rightarrow V'_\mu} W_\mu = 0, \quad (38)$$

and then  $V'_\mu = V_S$ . By virtue of the above considerations one can formulate the conclusion as follows:

**Theorem:**

Let us consider solutions of equations  $\frac{dr_\mu}{dV}$  and  $\frac{dm_\mu}{dV}$ ,  $r_\mu(V)$  and  $m_\mu(V)$  which exist on the interval  $(V_\mu, V_S]$ . Suppose that on the interval in question  $p(V)$  is finite and  $W_\mu$  is greater than zero. Moreover on the considered interval the following inequalities hold  $r_\mu > (2 m_\mu)^{\frac{1}{n-3}}$ , and  $m_\mu > -\frac{8\pi}{(n-2)} p r_\mu^{n-1}$ . Then, if  $\lim_{V \rightarrow V_\mu} p = p_\mu$  we arrive at the relations  $\sup_{(V_\mu, V_S)} \left( \frac{2 m_\mu}{r_\mu^{n-3}} \right) < 1$ , and  $\lim_{V \rightarrow V_\mu} W_\mu = 0$ .

Furthermore, it happened that the following relation is non-decreasing function of  $V$  on the interval  $(V_\mu, V_S]$

$$m_\mu - \frac{8 \pi}{(n-1)(n-2)} \rho r_\mu^{n-1}. \quad (39)$$

It can be readily seen by rearranging (10) and (11) to the form provided by

$$\frac{d}{dV} \left[ m_\mu - \frac{8 \pi}{(n-1)(n-2)} \rho r_\mu^{n-1} \right] = -\frac{8 \pi}{(n-1)(n-2)} \rho r_\mu^{n-1} \frac{d\rho}{dV}. \quad (40)$$

The right-hand side of the above equation is non-negative in the view of (4) and the monotonicity of the equation of state for the considered case. Further, we can arrange equation containing  $\frac{dW_\mu}{dV}$  in the form which yields

$$\frac{dW_\mu}{dV} - \frac{16 \pi V}{(n-1)(n-2)} \left[ (n-1) p + (n-3) \rho \right] = -\frac{2 (n-3) (n-2) V}{r_\mu^{n-1}} \left( m_\mu - \frac{8 \pi}{(n-1)(n-2)} \rho r_\mu^{n-1} \right). \quad (41)$$

One achieves that the quantity on the right-hand side, in the brackets, is non-decreasing. Because of the fact that  $r_\mu$  and  $V$  are positive, one draws a conclusion that the right-hand side of the above relation is non-negative on the interval in question.

Summing it all up, we established the monotonicity of the functions containing contributions of  $W_\mu$  and  $m_\mu$  for some

$V_i > V_\mu$ , i.e., on the interval  $(V_\mu, V_i]$ . Namely we have the following statement:

**Theorem:**

Let us consider equation  $m_\mu - \frac{8 \pi}{(n-1)(n-2)} \rho r_\mu^{n-1}$  which is a non-decreasing function of  $V$  on the interval  $(V_\mu, V_S]$ .

If the combination of  $W_\mu$  function of the form  $\frac{dW_\mu}{dV} - \frac{16 \pi V}{(n-1)(n-2)} \left[ (n-1) p + (n-3) \rho \right]$  is non-negative for some  $V_i > V_\mu$ , then it is non-negative for all  $V$  belonging to interval  $(V_\mu, V_i]$ .

As in four-dimensional case, the solution of Eqs. (10) and (11) for the case when  $W_\mu$  vanishes, can be split into two categories. Namely, one can tell about a regular zero of  $W_\mu$ , when  $\lim_{V \rightarrow V_\mu} r_\mu = 0$ , and an irregular zero of  $W_\mu$ , when  $\lim_{V \rightarrow V_\mu} r_\mu > 0$ . In order to determine the additional properties of the zeros of  $W_\mu$  function, let us consider the first case when  $\lim_{V \rightarrow V_\mu} W_\mu = 0$  and  $\lim_{V \rightarrow V_\mu} r_\mu = 0$ ,  $\lim_{V \rightarrow V_\mu} p_\mu < \infty$ . We use Eq.(40) and take the limit of both sides of it, when  $V \rightarrow V_\mu$ . Having in mind that

$$\lim_{V \rightarrow V_\mu} \left( \frac{m_\mu}{r_\mu^{n-1}} \right) = \lim_{V \rightarrow V_\mu} \frac{\frac{dm_\mu}{dr_\mu}}{(n-1) r_\mu^{n-2}} = \frac{8 \pi \rho}{(n-1)(n-2)}, \quad (42)$$

we reach to the conclusion that the following relation is provided:

$$\lim_{V \rightarrow V_\mu} \left[ \frac{m_\mu}{r_\mu^{n-1}} - \frac{8 \pi \rho(V_\mu)}{(n-1)(n-2)} \right] = 0. \quad (43)$$

Hence it implies that the limit of  $\frac{dW_\mu}{dV}$  is given by

$$\lim_{V \rightarrow V_\mu} \frac{dW_\mu}{dV} = \frac{16 \pi V_\mu (n-3)}{(n-1)(n-2)} \left( \rho(V_\mu) + \frac{n-1}{n-3} p_\mu \right). \quad (44)$$

Now, let us proceed to the case when  $\lim_{V \rightarrow V_\mu} r_\mu > 0$ . In this case we have that  $\lim_{V \rightarrow V_\mu} m_\mu = -\frac{8 \pi}{(n-2)} p_\mu r_\mu^{n-1}$ . Summing it all up, we take the limit of relation (40). It will be provided by the following expression:

$$\lim_{V \rightarrow V_\mu} \frac{dW_\mu}{dV} = \frac{16 \pi V_\mu (n-3)}{(n-2)} \left( \rho(V_\mu) + \frac{1+(n-2)(n-3)}{n-3} p_\mu \right). \quad (45)$$

Finally, it is worth mentioning that  $W_\mu$  implies the second order differential equation on the interval  $(V_\mu, V_S]$ , where one has that  $W_\mu$  is greater than zero. The explicit form of it yields

$$\begin{aligned} & \frac{d}{dV} \left[ \frac{1}{V} \frac{dW_\mu}{dV} - \frac{16 \pi}{n-2} \left[ (n-1) p + (n-3) \rho \right] \right] \\ &= \frac{n-1}{2(n-2) W_\mu V} \left[ \frac{dW_\mu}{dV} - \frac{16 \pi V}{(n-1)(n-2)} \left[ (n-1) p + (n-3) \rho \right] \right] \times \\ & \quad \left[ \frac{dW_\mu}{dV} - \frac{16 \pi V}{(n-1)(n-2)} \left[ (n-1) p + (n-3) \rho \right] \right], \end{aligned} \quad (46)$$

which can be found by the direct computations.

### III. CONFORMAL FACTOR

Let us introduce the function  $\psi_\mu(V)$ , which will constitute a conformal factor for a spatial metric  $g_{ab}$ . Namely,  $\tilde{g}_{ab+} = \Omega_+^2 g_{ab}$ , where  $\Omega_+ = \psi_\mu^{\frac{2}{n-3}}$  inside the considered star and  $\psi_\mu(V) = \left(\frac{1+V}{2}\right)$  outside the star. Having in mind the relation derived in [35] and equation(10), it can be revealed that inside the star, i.e.,  $V \in (V_\mu, V_S]$  this function satisfies

$$\frac{d\psi_\mu}{dV} = \frac{(n-3) \psi_\mu}{2 \sqrt{W_\mu} r_\mu} \left( 1 - \sqrt{1 - \frac{2 m_\mu}{r_\mu^{n-3}}} \right). \quad (47)$$

On the other hand, in the exterior region adjacent to the star, when  $V \in [V_S, 1)$  one has

$$\psi_\mu(V_S) = \left( \frac{1 + V_S}{2} \right). \quad (48)$$

A tedious calculations reveal that one can get the second order ordinary differential equation for  $\psi_\mu(V)$ . It implies

$$\frac{d^2\psi_\mu(V)}{dV^2} + \frac{1}{W_\mu} \left[ -\frac{(n-3)}{4(n-2)} {}^{(g)}R \psi_\mu(V) + \left( \frac{d\psi_\mu(V)}{dV} \right) D_a D^a V \right] = 0. \quad (49)$$

Using relation (3) for the Ricci tensor we attain to the fact that  ${}^{(g)}R = 16\pi\rho$ . Thus, the above equation for  $\psi_\mu(V)$  reduces to the following:

$$\frac{d^2\psi_\mu(V)}{dV^2} + \frac{8\pi}{W_\mu} \left[ \frac{V}{(n-3)} \left( (n-3)\rho + (n-1)p \right) \frac{d\psi_\mu(V)}{dV} - \frac{(n-3)}{2(n-2)} \rho \psi_\mu(V) \right] = 0. \quad (50)$$

Our main aim will be to find the sign of the function  $\frac{d^2\psi_\mu(V)}{dV^2}$ . Let us define the function  $f_\mu$  in the form as

$$f_\mu = \frac{4\pi(n-3)}{n-2} \rho - \frac{8\pi V}{n-3} \left( (n-3)\rho + (n-1)p \right) \frac{1}{\psi_\mu} \frac{d\psi_\mu}{dV}. \quad (51)$$

Next we calculated  $\frac{df_\mu}{dV}$ . Relations (50) and (4) clearly imply that

$$\begin{aligned} 8\pi V \left( (n-3)\rho + (n-1)p \right) \frac{df_\mu}{dV} &= \frac{16\pi^2(n-3)}{(n-2)^2} \left[ \left( (n-3) + 4(n-2) \right) \rho^2 \right. \\ &\quad \left. - 2(n-1)(n-2)p(\rho+p)\kappa \right] \\ &+ (n-3)f_\mu^2 - f_\mu \left[ 8\pi(n-3)(\rho+p)\kappa + \frac{64\pi^2V^2}{(n-3)W_\mu} \left( (n-3)\rho + (n-1)p \right)^2 \right. \\ &\quad \left. - \frac{8\pi\rho}{n-2} \left( (n-3)\rho + (n-1)p \right) \right], \end{aligned} \quad (52)$$

where  $\kappa = \frac{dp}{d\rho}$ . One can observe that the function  $f_\mu$  was chosen in such a way that  $\frac{d^2\psi_\mu(V)}{dV^2} = \alpha f_\mu$ , where  $\alpha$  is a constant value. Thus, the sign of  $f_\mu$  will be crucial to establish the sign of the second derivative with respect to  $V$  from  $\psi_\mu$ . If this equation implies that  $f_\mu < 0$  for some  $(V_\mu, V_S]$ , we get that  $\frac{df_\mu}{dV} > 0$ , under the following condition:

$$\left( (n-3) + 4(n-2) \right) \rho^2 - 2(n-1)(n-2)p(\rho+p)\kappa \geq 0. \quad (53)$$

But  $\lim_{V \rightarrow V_\mu} f_\mu(V) = 0$ , when  $V_\mu$  is a regular zero and  $\lim_{V \rightarrow V_\mu} f_\mu(V) = \infty$  if one has the case of irregular zero for  $V_\mu$ . The last statement yields that  $f_\mu$  is not negative at  $V = V_\mu$  and it is impossible to be negative for larger values of  $V$  unless  $\frac{df_\mu}{dV} < 0$  there. In turn, this is in contradiction to the conclusion from the inspection of equation (52). Just, we have that  $f_\mu \geq 0$ , which in turn implies that  $\frac{d^2\psi_\mu}{dV^2} \geq 0$ .

Thus, we arrive at the following conclusion:

**Theorem:**

Let us assume that  $\rho = \rho(p)$  is nonnegative, not decreasing function of pressure, satisfying the additional condition (53). It turns out that the function  $\psi_\mu$  defined by the relation  $\psi_\mu = 1/2(1+V)$  and its derivative with respect to  $V$  defined by (47) provides that  $\frac{d^2\psi_\mu}{dV^2} \geq 0$ .

#### IV. NONEXISTENCE OF PERFECT FLUID MATTER IN STATIC N-DIMENSIONAL BLACK HOLE BACKGROUND

The basic idea in our treatment of the problem in question will be to use the positive mass theorem in Bartnik formulation [43]. As in the proofs of the uniqueness of static  $n$ -dimensional black holes, we assume the validity of the

positive mass theorem which states that if one considers asymptotically flat, complete, orientable Riemannian manifold with non-negative scalar Ricci curvature and vanishing mass, then the considered manifold is isometric to  $(E^{n-1}, \delta_{ij})$  manifold.

To begin with, let us consider two conformal transformations provided by

$$\tilde{g}_{ab\pm} = \Omega_{\pm}^2 g_{ab}, \quad (54)$$

where  $\Omega_+$  is given by relations from the later section and  $\Omega_- = \xi_{\mu}^{\frac{2}{n-3}}$ , where  $\xi_{\mu} = \frac{1}{2}(1-V)$ . Hence we have two manifolds  $(\Sigma_{\pm}, \tilde{g}_{ij\pm})$ . Because of this fact we take into account two copies of the hypersurface  $\Sigma_+$  and  $\Sigma_-$  and define metric on them. By  $\partial_{nodeg}\Sigma$  one denotes all the components of the boundary of  $\Sigma$  which correspond to non-degenerate components of the black hole event horizons. One gets the following relations:

$$\Sigma_+ = \Sigma, \quad \tilde{g}_{ij+}, \quad (55)$$

$$\Sigma_- = \Sigma \cup \{p_i\}, \quad \tilde{g}_{ij-}, \quad (56)$$

where  $\Sigma \cup \{p_i\}$  describes a one point compactification of all asymptotically flat regions of the hypersurface  $\Sigma$ . Moreover,  $p_i$  denotes a point of the adequate asymptotically flat region.

It should be pointed out that the positive mass theorem [43] can not be implemented for  $(\Sigma_{\pm}, \tilde{g}_{ij\pm})$ . In order to fulfill the requirements of the aforementioned theorem we have to have the Riemannian manifold composed of two copies of  $\Sigma_{\pm}$ . We obtain  $\hat{\Sigma} = \Sigma_+ \cup \Sigma_+ \cup \partial_{nodeg}\Sigma$ . The differential structure on  $\hat{\Sigma}$  is provided by gluing  $\bar{\Sigma}_+ = \Sigma_+ \cup \partial_{nodeg}\Sigma$  with  $\bar{\Sigma}_- = \Sigma_- \cup \partial_{nodeg}\Sigma$ , and identifying  $\partial_{nodeg}\Sigma$  considered as a subset of  $\bar{\Sigma}_+$  with  $\partial_{nodeg}\Sigma$  considered as a subset of  $\bar{\Sigma}_-$ , using the identity map. The metric tensor defined on  $\Sigma_+ \cup \Sigma_-$  can be extended by continuity to smooth metric on  $\hat{\Sigma}$ . It implies

$$\hat{g}_{ij} |_{\Sigma_+} = \tilde{g}_{ij+} \quad \hat{g}_{ij} |_{\Sigma_-} = \tilde{g}_{ij-}. \quad (57)$$

In the case when  $\partial_{nodeg}\Sigma = 0$ , we arrive at  $\hat{\Sigma} = \Sigma$ ,  $\hat{g}_{ij} = \tilde{g}_{ij}$ .

Then, on the hypersurface  $\Sigma_+$  we have the asymptotical behaviour of the metric tensor  $\tilde{g}_{ij+}$  of the form

$$\tilde{g}_{ij+} = \delta_{ij} + \mathcal{O}\left(\frac{1}{r^{n-2}}\right), \quad (58)$$

while on  $\Sigma_-$  hypersurface, the asymptotic behaviour of the metric  $\tilde{g}_{ij-}$  yields

$$\tilde{g}_{ij-} = \left(\frac{m_{\mu}}{2r^{n-3}}\right)^{\frac{4}{n-3}} g_{ij} + \mathcal{O}\left(\frac{1}{r^{2n-3}}\right). \quad (59)$$

Let us examine the conformally rescaled Ricci curvature scalar  $\tilde{R}$  on both hypersurfaces in question. In the case of  $\Sigma_-$  hypersurface we get the following expression:

$$\tilde{R}(\Omega_-^2 g_{ab}) = 8\pi \left(\frac{1-V}{2}\right)^{-\frac{n+1}{n-3}} \left[ \rho \left(V+1\right) + 2 \left(\frac{n-1}{n-3}\right) V p \right]. \quad (60)$$

On the other hand, on  $\Sigma_+$  hypersurface,  $\tilde{R}(\Omega_+^2 g_{ab})$  may be written in the form as

$$\tilde{R}(\Omega_+^2 g_{ab}) = \left(\tilde{W} - W\right) \frac{4(n-2)}{n-3} \frac{1}{\psi_{\mu}^{\frac{n+1}{n-3}}} \frac{d^2\psi_{\mu}}{dV^2}, \quad (61)$$

where we have denoted  $W = D_i V D^i V$ . We remark that having in mind the asymptotical value of  $V$ , the asymptotic behaviour of  $W$  can be deduced. It is given by the expression

$$W = \frac{(n-3)^2 m_{\mu}^2}{r^{2(n-2)}} + \mathcal{O}\left(\frac{1}{r^{2n-3}}\right). \quad (62)$$

Because of the fact that  $\frac{d^2\psi_{\mu}}{dV^2} \geq 0$ , in order to have the non-negative conformally rescaled Ricci scalar tensor we assume that  $(\tilde{W} - W) \geq 0$ , where  $\tilde{W} = W_{\mu}$ .

Let us give some remarks concerning the above inequality, which will be crucial in what follows. In four-dimensional case [39] the proof of the aforementioned inequality is based on the identity due to Lindblom [38] which consists of the square of the Bach-Cotton tensor. It is well known that on three-dimensional manifold the Weyl tensor vanishes and the conformal properties of three-dimensional manifold are described by the Bach-Cotton tensor. In  $n$ -dimensional case one can generalize this idea and use the Bach-Cotton tensor in  $(n-1)$ -dimensional spacetime. Then the boundary point maximum principle to prove the inequality can be implemented [41].

On this account it yields [44]

$$\begin{aligned} {}^{(g)}C_{ijk} {}^{(g)}C^{ijk} &= \alpha \frac{W \nabla_k \nabla^k W}{2 V^4} - \frac{8 \pi \alpha W}{V^3} \left( \frac{2n-5}{n-2} \right) D_i \rho D^i V + \frac{16 \pi \alpha W^2 \rho}{(n-2) V^4} \\ &+ \frac{8 \pi (n-2)}{V^3 (n-3)^2} D_i V D^i W \left[ (n-2) p + (n-3) \rho \right] - \frac{(n-2)}{4 V^4 (n-3)^2} D_a W D^a W \\ &+ \frac{64 \pi^2 W}{V^2 (n-2)^2 (n-3)^2} \left[ \alpha_1 p^2 + \alpha_2 p \rho + \alpha_3 \rho^2 \right], \end{aligned} \quad (63)$$

where  $\alpha, \alpha_1, \alpha_2, \alpha_3$  are constant coefficients [44]. Next, the maximum principle can be used in the proof. What is more, one can also investigate the identity including the Weyl tensor describing the conformal properties in  $n > 3$  dimensions. After tedious computations it can be revealed that the square of the  $(n-1)$ -dimensional Weyl tensor implies

$$\begin{aligned} {}^{(g)}C_{ijkl} {}^{(g)}C^{ijkl} &= \left( \frac{n-2}{n-3} \right)^2 \left[ 64 \pi^2 p^2 (n-1) - 128 \pi^2 p \left[ (n-1) p + (n-3) \rho \right] \right] \\ &+ \frac{64 \pi^2 (n-3)}{(n-2)^2} \left[ (n-1) p + (n-3) \rho \right]^2 \\ &- \frac{4}{n-2} \left[ \frac{64 \pi^2 (n-1)}{n-2} (\rho - p)^2 + \frac{128 \pi^2}{(n-2)^2} (\rho - p) \left[ (n-1) p + (n-3) \rho \right] \right] \\ &+ \frac{8 (n-1)}{(n-2) (n-3)^2} 256 \pi^2 \rho^2 \\ &- \frac{4 (n-2)^3}{V^2 (n-3)^2 (n-2)} \left[ \frac{1}{2} \nabla_k \nabla^k W + \frac{16 \pi}{n-2} W \rho - \left( \frac{2n-5}{n-2} \right) D_i \rho D^i V \right] \end{aligned} \quad (64)$$

In accordance with the above, the maximum principle should be used to prove the inequality in question.

Returning to the problem in question, one has that both Ricci scalar curvature tensors are non-negative and moreover equation (58) implies that the total ADM mass also vanishes. As a consequence of the positive mass theorem the hypersurface  $\bar{\Sigma}$  is isometric to flat manifold or in the other words, the Cauchy surface in question is conformally flat. Consequently, we have that

$$\tilde{R}(\Omega_-^2 g_{ab}) = \tilde{R}(\Omega_+^2 g_{ab}) = 0. \quad (65)$$

The above relation provides the following:

$$W = W_\mu, \quad \rho = p = 0. \quad (66)$$

The second condition yields that the spacetime under consideration ought to be empty, i.e., one excludes any static configuration composed of  $n$ -dimensional black hole and a perfect fluid star.

Finally, the above considerations enable us to formulate the main conclusion of our work.

**Theorem:**

Let us consider a static black hole spacetime with an asymptotically timelike Killing vector field  $k_\mu$ . Assume that one considers in such background a perfect fluid star which has the surface of the level surface set  $\{V = V_S > 0\}$ . Suppose moreover that equation of state  $p = p(\rho)$  is provided by

$$\left( (n-3) + 4 (n-2) \right) \rho^2 - 2 (n-1)(n-2) p (\rho + p) \kappa \geq 0,$$

and the inequality  $(W - W_\mu)_{V=0} \leq 0$ , where  $W_\mu$  is defined by relation (14) and  $W = D_j V D^j V$ , is satisfied on the black hole event horizon. Then under the above conditions, a perfect fluid star cannot exist in a static  $n$ -dimensional

black hole spacetime.

Now, we proceed to give some remarks concerning spherical symmetry of the general static relativistic  $n$ -dimensional perfect fluid stellar model. The arguments that lead us to state that the spatial geometry  $g_{ab}$  of a static stellar model described by equations (2)-(4) has spherical symmetry are mainly the same as in the uniqueness proof of static  $n$ -dimensional black holes [20]. We outline the crucial steps of them.

To begin with, one can choose  $V$  as a local coordinate in the neighborhood  $U \in \Sigma$ . With this understanding the metric on the hypersurface  $\Sigma$  can be written in the form as

$$ds^2 = \eta^2 dV^2 + h_{AB} dx^A dx^B, \quad (67)$$

where we set  $\eta^2 = (D_m V D^m V)^2$ . The  $x_A$ -coordinates are chosen in such a way that their trajectories are orthogonal to each of the level set. One can also rewrite  $g_{ij}$  in a conformally flat form [20]

$$g_{ij} = \mathcal{U}^{\frac{1}{n-3}} \delta_{ij}, \quad (68)$$

where we have defined a smooth function on  $(\hat{\Sigma}, \delta_{ij})$ , namely  $\mathcal{U} = \frac{2}{1+V}$ . It can be easily found that Einstein equations of motion reduces to the Laplace equation on the  $(n-1)$  Euclidean manifold  $\nabla_i \nabla^i \mathcal{U} = 0$ , where  $\nabla$  is the connection on a flat manifold. Accordingly, we can adopt for the metric  $\delta_{ij}$  in the flat base space the following metric:

$$\delta_{ij} dx^i dx^j = \tilde{\eta}^2 d\mathcal{U}^2 + \tilde{h}_{AB} dx^A dx^B. \quad (69)$$

Further, one can show that the embedding of the stellar surface into the Euclidean  $(n-1)$ -dimensional space is totally umbilical [45]. This embedding have to be hyperspherical, i.e., each of the connected components of the surface in question is a geometric sphere with a certain radius determined by the value of  $\eta|_{surf}$ . One can always locate one of the connected component of the stellar surface at  $\eta = \eta_0$  without loss of generality. It turns out that, we have to do with a boundary value problem for the Laplace equation on the base space  $\Omega = E^{n-1}/B^{n-1}$  with the Dirichlet boundary condition  $\mathcal{U}|_{surf}$  and the asymptotic decay condition  $\mathcal{U} = 1 + \mathcal{O}\left(\frac{1}{r^{n-3}}\right)$ .

## V. INEQUALITY

In this section we restrict our attention to the physical meaning of our hypothesis. On this account, let us consider the case when  $(W - W_\mu)_{V=0} \leq 0$ . To begin with, we compute the surface gravity  $\kappa$  of  $n$ -dimensional static black hole. It yields the following relation:

$$\kappa = \sqrt{-\frac{1}{4} g^{tt} g^{rr}(g_{tt,r})} |_{r=r_{BH}} = \frac{(n-3) m_{BH}^{3-n}}{2^{n-2}}. \quad (70)$$

Then, letting  $V = 0$  and having in mind the relation for  $W_\mu$ , we receive

$$\sqrt{W_{V=0}} = \frac{n-3}{2^{\frac{n-2}{n-3}} \mu^{\frac{1}{n-3}}}. \quad (71)$$

The Killing vectors for static asymptotically flat spacetime may be used to find the coordinate independent expression for black hole mass. By virtue of it, we can readily verify that the following expression is satisfied:

$$\mu^{\frac{1}{n-3}} \leq \frac{1}{\left(2^{\frac{n-2}{n-3}} \kappa\right)^{\frac{1}{n-3}}} = \left[ \frac{(n-2) A_{BH}}{8 \pi 2^{n-2} M_{BH}} \right]^{\frac{1}{(n-3)^2}}. \quad (72)$$

Moreover, having in mind that [24]

$$\frac{n-3}{n-2} M_{BH} = \frac{\kappa A_{BH}}{8 \pi}, \quad (73)$$

one arrives at the relation provided by

$$\mu^{\frac{1}{n-3}} \leq (m_{BH})^{\frac{1}{n-3}}, \quad (74)$$

where black hole mass yields

$$M_{BH} = \frac{(n-2)}{8\pi} \Omega_{(n-2)} m_{BH}. \quad (75)$$

In order to establish the upper limit on  $m_\mu$ , we consider the next inequality, which takes place on the surface of the perfect fluid star  $(W - W_\mu)_{V=V_S} \leq 0$ . In the case under consideration we get

$$\sqrt{W_\mu} = \frac{(n-3) (1-V_S)^{\frac{n-2}{n-3}}}{2^{\frac{n-2}{n-3}} \mu^{\frac{1}{n-3}}}. \quad (76)$$

Let us take the asymptotic value of  $\sqrt{W_\mu}$ , on the left-hand side of the above equation

$$m_\mu \leq \frac{\left(1 - V_S^2\right)^{\frac{n-2}{n-3}}}{2^{\frac{n-2}{n-3}} \mu^{\frac{1}{n-3}}} r_\mu^{n-2}. \quad (77)$$

Integrating over the  $(n-2)$ -dimensional sphere we get the expression of the form

$$\mu^{\frac{1}{n-3}} \leq \frac{\left(1 - V_S^2\right)^{\frac{n-2}{n-3}} A_{star}}{8\pi 2^{\frac{n-2}{n-3}} M_{star}}, \quad (78)$$

where  $A_{star} = \Omega_{(n-2)} r_S^{n-2}$  and  $M_{star} = \frac{n-2}{8\pi} \Omega_{(n-2)} m_\mu$ , are the area of the star surface and its mass, respectively. Hence, we can readily write down

$$\mu^{\frac{1}{n-3}} \leq (m_\mu)^{\frac{1}{n-3}}. \quad (79)$$

On this account, it is customary to conclude that

$$(m_\mu)^{\frac{1}{n-3}} \leq (m_{BH})^{\frac{1}{n-3}}. \quad (80)$$

Taking into account the inequality (80) we can assert that our main theorem states that a perfect fluid star of a smaller mass than a black hole mass in not able to exist outside a static black hole spacetime.

## VI. CONCLUSIONS

In our paper we have considered matter fields of perfect fluid in the spacetime of  $n$ -dimensional static asymptotically flat black hole. The main aim was to prove the uniqueness of such a configuration. As to how this might be done, can be inferred from the generalization of the method using the positive mass theorem [43] for showing the uniqueness of static black holes. Namely, considering the conformal transformation on the hypersurface  $t = const$ , one finds that the Ricci scalar curvature tensor is non-negative and moreover  $\Sigma$  becomes a hypersurface with zero ADM mass. As a consequence, the manifold in question satisfies the conditions to apply the positive mass theorem, which in turn yields that  $\Sigma$  is isometric to flat manifold. Then, the inspection of the conformally rescaled Ricci scalar tensors enables one to find that  $\rho = p = 0$ . It means that we excluded any static configurations composed of  $n$ -dimensional black hole and a perfect fluid star. The aforementioned proof was achieved under the auxiliary inequality, which means that the mass of the perfect fluid star is smaller than the mass of static  $n$ -dimensional black hole.

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